GLOBAL ATTRACTORS FOR STRONGLY DAMPED WAVE EQUATIONS WITH DISPLACEMENT DEPENDENT DAMPING AND NONLINEAR SOURCE TERM OF CRITICAL EXPONENT

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Abstract. In this paper the long time behaviour of the solutions of the 3-D strongly damped wave equation is studied. It is shown that the semigroup generated by this equation possesses a global attractor in $H_0^1(\Omega)\times L_2(\Omega)$ and then it is proved that this is also a global attractor in $(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)$.

1. Introduction

We consider the following initial-boundary value problem for the strongly damped wave equation:

$$w_{tt}-\Delta w_t+\sigma(w)w_t-\Delta w+f(w)=g(x) \qquad \text{in } (0,\infty)\times\Omega, \tag{1.1}$$

$$w=0 \qquad \text{on } (0,\infty)\times\partial\Omega, \tag{1.2}$$

$$w(0,\cdot)=w_0\ , \qquad w_t(0,\cdot)=w_1 \qquad \text{in } \Omega, \tag{1.3}$$

where $\Omega\subset R^3$ is a bounded domain with sufficiently smooth boundary and $g\in L_2(\Omega)$.

As shown in [6] and [13], equation (1.1) is related to the following reaction-diffusion equation with memory:

$$w_t(t,x)=\int\limits_{-\infty}^{t}K(t,s)\Delta w(s,x)ds-f(w(t,x))+g(x). \tag{1.4}$$

Namely, if $K(t,s)=\frac{1-\alpha}{\lambda}e^{-\frac{t-s}{\lambda}}+2\alpha\delta(t-s)$ then (1.4) can be transformed into

$$\lambda w_{tt}-\alpha\lambda\Delta w_t+(1+\lambda f'(w))w_t-\Delta w+f(w)=g,$$

where $\lambda>0$, $\alpha\in[0,1)$ and δ is a Dirac delta function. This equation is interesting from a physical viewpoint as a model describing the flow of viscoelastic fluids (see [6] and [13] for details).

When $\sigma(\cdot)\equiv 0$ the equation (1.1) becomes

$$w_{tt}-\Delta w_t-\Delta w+f(w)=g. \tag{1.5}$$

The long time behaviour (in terms of attractors) of solutions in this case has been studied by many authors (see [2], [5], [7], [14], [15], [19], [22] and references therein). In [14] the existence of a global attractor for (1.5) with critical source term (i.e. in the case when the growth of f is of order 5) was proved. However, the regularity of the global attractor in that article was established only in the subcritical case. For

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the critical case, the regularity of the global attractor of (1.5) was proved in [15], under the assumptions

$$f \in C^1(R),\ |f'(s)| \le c(1+|s|^4),\ \forall s \in R \text{ and } \liminf_{|s|\to\infty} f'(s) > -\lambda_1 \tag{1.6}$$

or

$$f \in C^2(R),\ |f''(s)| \le c(1+|s|^3),\ \forall s \in R \text{ and } \liminf_{|s|\to\infty} \frac{f(s)}{s} > -\lambda_1, \tag{1.7}$$

where λ_1 is a first eigenvalue of $-\Delta$ with zero Dirichlet data. In that article the authors obtained a regular estimate for w_{tt} (when $w(t,x)$ is a weak solution of (1.5)) and then proved the asymptotic regularity of the solution of the non-autonomous equation

$$-\Delta w_t - \Delta w + f(w) = g - w_{tt}.$$

In [5] and [19], the regularity of the global attractor of (1.5) was proved under the following weaker condition on the source term:

$$f \in C(R),\ |f(u)-f(v)| \le c(1+|u|^4+|v|^4)|u-v|,\ \forall u,v \in R \text{ and } \liminf_{|s|\to\infty} \frac{f(s)}{s} > -\lambda_1.$$

In [8], the authors investigated the weak attractor for the quasi-linear strongly damped equation

$$w_{tt} - \Delta w_t - \Delta w + f(w) = \nabla \cdot \varphi'(\nabla w) + g$$

under the following conditions on the nonlinear functions f and φ:

$$f \in C^1(R),\quad -C + a_1 |s|^q \le f'(s) \le C|s|^q,\ \forall s \in R,$$

$$\varphi \in C^2(R^3,R),\ \ a_2 |\eta|^{p-1} |\xi|^2 \le \sum_{i,j=1}^{3} \frac{\partial^2 \varphi(\eta)}{\partial \eta_i \partial \eta_j} \xi_i \xi_j \le a_3(1+|\eta|^{p-1})|\xi|^2,\ \ \forall \xi,\eta \in R^3,$$

for some $a_i > 0$, $(i = 1,2,3)$, $C > 0$, $q > 0$ and $p \in [1,5)$. When $\frac{\partial^2 \varphi}{\partial \eta_i \partial \eta_j} = 0$, $(i,j = 1,2,3)$, the strong attractor has also been studied. Recently, in [3], the authors have studied the global attractor for the strongly damped abstract equation

$$w_{tt} + D(w,w_t) + Aw + F(w) = 0.$$

However, the approaches of the articles mentioned above, in general, do not seem to be applicable to (1.1). The difficulty is caused by the term $\sigma(w)w_t$, when the function $\sigma(\cdot)$ is not differentiable and the growth condition imposed on $\sigma(\cdot)$ is critical. In this paper we prove the existence of the global attractors for (1.1)-(1.3) in $H_0^1(\Omega) \times L_2(\Omega)$ and $(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$. Then using the embedding $H^{\frac{3}{2}+\varepsilon}(\Omega) \subset C(\overline{\Omega})$ we show that these attractors coincide.

2. Well-posedness and the statement of the main result

We start with the conditions on nonlinear terms f and σ.

- $f \in C(R),\ \ |f(s) - f(t)| \le c(1+|s|^4+|t|^4)|s-t|,\ \forall s,t \in R,$ (2.1)
- $\liminf_{|s|\to\infty} \frac{f(s)}{s} > -\lambda_1,$ where $\lambda_1 = \inf_{\varphi \in H_0^1(\Omega), \varphi \ne 0} \frac{\|\nabla \varphi\|^2_{L_2(\Omega)}}{\|\varphi\|^2_{L_2(\Omega)}}$, (2.2)
- $\sigma \in C(R),\ \ \sigma(s) \ge 0,\ \ |\sigma(s)| \le c(1+|s|^4),\ \ \forall s \in R.$ (2.3)

By the standard Galerkin's method it is easy to prove the following existence theorem:

Theorem 2.1. *Let conditions (2.1)-(2.3) hold. Then for every $T > 0$ and every $(w_0, w_1) \in \mathcal{H} := H_0^1(\Omega) \times L_2(\Omega)$, the problem (1.1)-(1.3) admits a weak solution*

$$w \in C([0,T]; H_0^1(\Omega)),\ w_t \in C([0,T]; L_2(\Omega)) \cap L_2(0,T; H_0^1(\Omega)),$$

which satisfies the following energy equality

$$E(w(t)) + \int_s^t \|\nabla w_t(\tau)\|^2_{L_2(\Omega)}\, d\tau + \int_s^t \langle \sigma(w(\tau)) w_t(\tau), w_t(\tau)\rangle\, d\tau + \langle F(w(t)), 1\rangle -$$

$$- \langle g, w(t)\rangle = E(w(s)) + \langle F(w(s)), 1\rangle - \langle g, w(s)\rangle\,, \qquad 0 \le s \le t \le T, \qquad (2.4)$$

where $E(w(t)) = \frac{1}{2}(\|\nabla w(t)\|^2_{L_2(\Omega)} + \|w_t(t)\|^2_{L_2(\Omega)})$, $\langle u, v\rangle = \int\limits_\Omega u(x)v(x)dx$ *and* $F(w) = \int\limits_0^w f(u)du$.

Now using the method of [16, Proposition 2.2] let us prove the following uniqueness theorem:

Theorem 2.2. *Let conditions (2.1)-(2.3) hold. If $w(t,\cdot)$ and $\widehat{w}(t,\cdot)$ are the weak solutions of (1.1)-(1.3), determined by Theorem 2.1, with initial data (w_0, w_1) and $(\widehat{w}_0, \widehat{w}_1)$ respectively, then*

$$\|w(T) - \widehat{w}(T)\|^2_{H^1(\Omega)} + \|w_t(T) - \widehat{w}_t(T)\|^2_{H^{-1}(\Omega)} \le$$

$$\le c(T,R)\left(\|w_0 - \widehat{w}_0\|_{H^1(\Omega)} + \|w_1 - \widehat{w}_1\|_{H^{-1}(\Omega)}\right)$$

where $c : R_+ \times R_+ \to R_+$ is a nondecreasing function with respect to each variable and $R = \max\{\|(w_0, w_1)\|_{\mathcal{H}}, \|(\widehat{w}_0, \widehat{w}_1)\|_{\mathcal{H}}\}$.

Proof. By (2.1)-(2.4), it follows that

$$\|(w(t), w_t(t))\|_{\mathcal{H}} + \|(\widehat{w}(t), \widehat{w}_t(t))\|_{\mathcal{H}} \le c_1(R), \quad \forall t \ge 0.$$

Denote $u(t,\cdot) = w(t,\cdot) - \widehat{w}(t,\cdot)$ and $\widehat{u}(t,\cdot) = \int\limits_0^t u(\tau,\cdot)d\tau$. Integrating (1.1) for $w(t,\cdot)$ and $\widehat{w}(t,\cdot)$ on $[0,t]$ and taking the difference, we have

$$u_t - \Delta u + \Sigma(w) - \Sigma(\widehat{w}) - \Delta\widehat{u} + \int_0^t \left(f(w(\tau,)) - f(\widehat{w}(\tau,))\right) d\tau =$$

$$= \Sigma(w_0) - \Sigma(\widehat{w}_0) - \Delta(w_0 - \widehat{w}_0) + w_1 - \widehat{w}_1, \qquad \forall t \ge 0, \qquad (2.5)$$

where $\Sigma(w) = \int\limits_0^w \sigma(s)ds$. Testing (2.5) by u and taking into account (2.1), (2.3), (2.4) and monotonicity of $\Sigma(\cdot)$, we find

$$\frac{d}{dt}E(\widehat{u}(t)) + \frac{1}{2}\|\nabla u(t)\|^2_{L_2(\Omega)} \le$$

$$\le c_2(R)\left(\|\nabla(w_0 - \widehat{w}_0)\|^2_{L_2(\Omega)} + \|w_1 - \widehat{w}_1\|^2_{H^{-1}(\Omega)}\right) +$$

$$+ c_2(R)t\int_0^t \|\nabla u(\tau)\|^2_{L_2(\Omega)}\, d\tau, \quad \forall t \ge 0 \qquad (2.6)$$

and consequently

$$\frac{d}{dt}\widehat{E}(\widehat{u}(t)) \le c_2(R)\left(\|w_0-\widehat{w}_0\|^2_{H^1(\Omega)}+\|w_1-\widehat{w}_1\|^2_{H^{-1}(\Omega)}\right)+2c_2(R)t\widehat{E}(\widehat{u}(t)),$$

where $\widehat{E}(\widehat{u}(t)) = E(\widehat{u}(t)) + \frac{1}{2}\int\limits_0^t \|\nabla u(\tau)\|^2_{L_2(\Omega)}\,d\tau$. Applying Gronwall's lemma to the last inequality, we get

$$\widehat{E}(\widehat{u}(t)) \le c_3(R)e^{c_2(R)t^2}\left(\|w_0-\widehat{w}_0\|^2_{H^1(\Omega)}+\|w_1-\widehat{w}_1\|^2_{H^{-1}(\Omega)}\right) \tag{2.7}$$

By (2.1), (2.3), (2.4) and (2.7), it follows that

$$\left|\frac{d}{dt}E(\widehat{u}(t))\right| \le |\langle u_t(t),u(t)\rangle| + |\langle \nabla\widehat{u}(t),\nabla u(t)\rangle| \le$$

$$\le c_4(R)\left(\|u(t)\|_{L_2(\Omega)}+\|\nabla\widehat{u}(t)\|_{L_2(\Omega)}\right) \le$$

$$\le c_5(R)e^{\frac{c_2(R)t^2}{2}}\left(\|w_0-\widehat{w}_0\|_{H^1(\Omega)}+\|w_1-\widehat{w}_1\|_{H^{-1}(\Omega)}\right), \quad \forall t\ge 0.$$

Taking into account (2.7) and the last inequality in (2.6), we obtain

$$\|\nabla u(t)\|^2_{L_2(\Omega)} \le c_6(R)(1+t)e^{c_2(R)t^2}\left(\|w_0-\widehat{w}_0\|_{H^1(\Omega)}+\right.$$

$$\left.+\|w_1-\widehat{w}_1\|_{H^{-1}(\Omega)}\right), \quad \forall t\ge 0.$$

Now, from (2.5), we have

$$\|u_t(t)\|_{H^{-1}(\Omega)} \le \|\nabla u(t)\|_{L_2(\Omega)} + \|\nabla\widehat{u}(t)\|_{L_2(\Omega)} + \|\Sigma(w(t))-\Sigma(\widehat{w}(t))\|_{H^{-1}(\Omega)} +$$

$$+\int\limits_0^t \|f(w(\tau,))-f(\widehat{w}(\tau,))\|_{H^{-1}(\Omega)}\,d\tau + \|\Sigma(w_0)-\Sigma(\widehat{w}_0)\|_{H^{-1}(\Omega)} +$$

$$+\|\nabla(w_0-\widehat{w}_0)\|_{L_2(\Omega)} + \|w_1-\widehat{w}_1\|_{H^{-1}(\Omega)},$$

which due to the above inequalities gives

$$\|u_t(t)\|^2_{H^{-1}(\Omega)} \le c_7(R)(1+t)e^{c_2(R)t^2}\left(\|w_0-\widehat{w}_0\|_{H^1(\Omega)}+\right.$$

$$\left.+\|w_1-\widehat{w}_1\|_{H^{-1}(\Omega)}\right), \quad \forall t\ge 0.$$

□

Thus by Theorem 2.1 and Theorem 2.2, it follows that by the formula $S(t)(w_0,w_1) = (w(t),w_t(t))$, problem (1.1)-(1.3) generates a weakly continuous (in the sense, if $\varphi_n \to \varphi$ strongly then $S(t)\varphi_n \to S(t)\varphi$ weakly) semigroup $\{S(t)\}_{t\ge 0}$ in $\mathcal{H}$, where $w(t,\cdot)$ is a weak solution of (1.1)-(1.3), determined by Theorem 2.1, with initial data (w_0,w_1). To show the strong continuity of $\{S(t)\}_{t\ge 0}$ we firstly prove the following lemma:

Lemma 2.1. *Let $\varphi \in C(R)$ and $|\varphi(x)| \le c(1+|x|^r)$ for every $x\in R$ and some $r\ge 1$. If $v_n \to v$ strongly in $L_q(\Omega)$ for $q\ge r$, then $\varphi(v_n)\to\varphi(v)$ strongly in $L_{\frac{q}{r}}(\Omega)$.*

Proof. By the assumption of the lemma, there exists a subsequence $\{v_{n_k}\}$ such that $v_{n_k} \to v$ a.e. in Ω. Then by Egorov's theorem, for any $\varepsilon > 0$ there exists a subset $A_\varepsilon \subset \Omega$ such that $mes(A_\varepsilon) < \varepsilon$ and $v_{n_k} \to v$ uniformly in $\Omega \backslash A_\varepsilon$. Hence for large enough k

$$|v_{n_k}(x)| \leq 1 + |v(x)| \quad \text{in } \Omega \backslash A_\varepsilon$$

and consequently

$$|\varphi(v_{n_k}(x))| \leq c_1(1 + |v(x)|^r) \ \text{ in } \Omega \backslash A_\varepsilon.$$

Applying Lebesgue's theorem we get

$$\lim_{k\to\infty} \|\varphi(v_{n_k}) - \varphi(v)\|_{L_{\frac{q}{r}}(\Omega \backslash A_\varepsilon)} = 0. \tag{2.8}$$

On the other hand since we have

$$\lim_{k\to\infty} \|v_{n_k}\|_{L_q(A_\varepsilon)} = \|v\|_{L_q(A_\varepsilon)},$$

the inequality

$$\limsup_{k\to\infty} \|\varphi(v_{n_k})\|^{\frac{q}{r}}_{L_{\frac{q}{r}}(A_\varepsilon)} < c_3(\varepsilon + \|v\|^q_{L_q(A_\varepsilon)})$$

is satisfied. The last inequality together with (2.8) implies that

$$\limsup_{k\to\infty} \|\varphi(v_{n_k}) - \varphi(v)\|^{\frac{q}{r}}_{L_{\frac{q}{r}}(\Omega)} \leq c_4 \lim_{\varepsilon\to 0}(\varepsilon + \|v\|^q_{L_q(A_\varepsilon)}) = 0.$$

□

Theorem 2.3. *Under conditions (2.1)-(2.3) the semigroup $\{S(t)\}_{t\geq 0}$ is strongly continuous in $\mathcal{H}$.*

Proof. Let $(w_{0n}, w_{1n}) \to (w_0, w_1)$ strongly in $\mathcal{H}$. Denoting $(w_n(t), w_{tn}(t)) = S(t)(w_{0n}, w_{1n})$, $(w(t), w_t(t)) = S(t)(w_0, w_1)$ and $u_n(t) = w_n(t) - w(t)$, by (1.1) we have

$$u_{ntt} - \Delta u_{nt} + \sigma(w_n)w_{nt} - \sigma(w)w_t - \Delta u_n + f(w_n(\tau)) - f(w(t)) = 0.$$

Since, by Theorem 2.1, every term of the above equation belongs to $L_2(0,T;H^{-1}(\Omega))$, testing it by u_{nt}, we obtain

$$E(u_n(t)) \leq E(u_n(0)) + c\,\|\sigma(w_n) - \sigma(w)\|^2_{C([0,T];L_{\frac{3}{2}}(\Omega))} + c\int_0^t E(u_n(s))ds,\ \forall t \in [0,T].$$

Applying Gronwall's lemma we have

$$E(u_n(T)) \leq \left(E(u_n(0)) + c\,\|\sigma(w_n) - \sigma(w)\|^2_{C([0,T];L_{\frac{3}{2}}(\Omega))}\right)e^{cT}, \ \ \forall T \geq 0. \tag{2.9}$$

By Theorem 2.2, it follows that

$$\lim_{n\to\infty} \|w_n - w\|_{C([0,T];L_6(\Omega))} = 0.$$

Now applying Lemma 2.1 it is easy to see that

$$\lim_{n\to\infty} \|\sigma(w_n) - \sigma(w)\|_{C([0,T];L_{\frac{3}{2}}(\Omega))} = 0,$$

which together with (2.9) yields that $S(T)(w_{0n}, w_{1n}) \to S(T)(w_0, w_1)$ strongly in $\mathcal{H}$, for every $T \geq 0$. □

Now let us recall the definition of a global attractor.

Definition ([17]). Let $\{V(t)\}_{t\geq 0}$ be a semigroup on a metric space $(X,\, d)$. A compact set $\mathcal{A}\subset X$ is called a global attractor for the semigroup $\{V(t)\}_{t\geq 0}$ iff

- $\mathcal{A}$ is invariant, i.e. $V(t)\mathcal{A}=\mathcal{A}$, $\forall t\geq 0$;
- $\lim\limits_{t\to\infty}\sup\limits_{v\in B}\inf\limits_{u\in\mathcal{A}} d(V(t)v,u)=0$ for each bounded set $B\subset X$.

Our main result is as follows:

Theorem 2.4. *Under the conditions (2.1)-(2.3), the semigroup $\{S(t)\}_{t\geq 0}$ generated by the problem (1.1)-(1.3) possesses a global attractor $\mathcal{A}$ in $\mathcal{H}$, which is also a global attractor in $\mathcal{H}_1:=(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)$.*

Remark 2.1. *We note that if the condition (2.3) is replaced by*

$$\sigma\in C(R),\;\; \sigma(s)\geq 0,\;\; |\sigma(s)|\leq c(1+|s|^p),\; 0\leq p<4,\; \forall s\in R,$$

then using the methods of [5] , [19] *and* [21] *one can prove Theorem 2.4. If we assume*

$$\sigma\in C^1(R),\;\; \sigma(s)\geq 0,\;\; |\sigma'(s)|\leq c(1+|s|),\; \forall s\in R,$$

instead of (2.3), then the method of [15] *can be applied to (1.1)-(1.3). In this case, as in* [20]*, one can show that a global attractor $\mathcal{A}$ attracts every bounded subset of $\mathcal{H}$ in the topology of $H_0^1(\Omega)\times H_0^1(\Omega)$.*

Remark 2.2. *We also note that problem (1.1)-(1.3), in 3-D case, without the strong damping $-\Delta w_t$ was considered in* [11] *and* [16]*. In this case, when $\sigma(\cdot)$ is not globally bounded, the existence of a global attractor in the strong topology of $\mathcal{H}$ and the regularity of the weak attractor remain open (see* [11] *and* [16] *for details).*

3. Existence of the global attractor in $\mathcal{H}$

We start with the following asymptotic compactness lemma:

Lemma 3.1. *Let conditions (2.1)-(2.3) hold and B be a bounded subset of $\mathcal{H}$. Then every sequence of the form $\{S(t_n)\varphi_n\}_{n=1}^{\infty}$, $\{\varphi_n\}_{n=1}^{\infty}\subset B$, $t_n\to\infty$, has a convergent subsequence in $\mathcal{H}$.*

Proof. By (2.4), we have

$$\left\{\begin{array}{l} \sup\limits_{t\geq 0}\sup\limits_{\varphi\in B}\|S(t)\varphi\|_{\mathcal{H}}<\infty, \\ \sup\limits_{\varphi\in B}\int\limits_0^{\infty}\|PS(t)\varphi\|^2_{H_0^1(\Omega)}\,dt<\infty, \end{array}\right. \tag{3.1}$$

where $P:\mathcal{H}\to L_2(\Omega)$ is a projection map, i.e. $P\varphi=\varphi_2$, for every $\varphi=(\varphi_1,\varphi_2)\in\mathcal{H}$. So for any $T_0\geq 1$ there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ such that $t_{n_k}\geq T_0$ and

$$\left\{\begin{array}{ll} w_k\to w & \text{weakly star in } L_\infty(0,\infty;H_0^1(\Omega)), \\ w_{kt}\to w_t & \text{weakly in } L_2(0,\infty;H_0^1(\Omega)), \end{array}\right. \tag{3.2}$$

for some $w\in L_\infty(0,\infty;H_0^1(\Omega))\cap W^{1,\infty}(0,\infty;L_2(\Omega))\cap W^{1,2}_{loc}(0,\infty;H_0^1(\Omega))$, where $(w_k(t),w_{kt}(t))=S(t+t_{n_k}-T_0)\varphi_{n_k}$. Now multiplying the equality

$$(w_k-w_m)_{tt}-\Delta(w_{kt}-w_{mt})+\sigma(w_k)w_{kt}-\sigma(w_m)w_{mt}-\Delta(w_k-w_m)+$$
$$+f(w_k)-f(w_m)=0$$

by $(w_{kt} - w_{mt} + \frac{\lambda_1}{2}(w_k - w_m))$ and integrating over $(s,T)\times\Omega$, we obtain

$$\frac{1}{2}E(w_k(T)-w_m(T))+\lambda_1\int_s^T E(w_k(t)-w_m(t))dt+$$

$$+\int_s^T \langle \sigma(w_k(t))w_{kt}(t)-\sigma(w_m(t))w_{mt}(t), w_{kt}(t)-w_{mt}(t)\rangle\, dt+$$

$$+\frac{\lambda_1}{2}\left\langle \widehat{\Sigma}(w_k(T))+\widehat{\Sigma}(w_m(T)),1\right\rangle-\frac{\lambda_1}{2}\int_s^T \langle \sigma(w_k(t))w_{kt}(t), w_m(t)\rangle\, dt$$

$$-\frac{\lambda_1}{2}\int_s^T \langle \sigma(w_m(t))w_{mt}(t), w_k(t)\rangle\, dt+\langle F(w_k(T))+F(w_m(T)),1\rangle-$$

$$-\int_s^T \langle f(w_k(t)), w_{mt}(t)\rangle\, dt-\int_s^T \langle f(w_m(t)), w_{kt}(t)\rangle\, dt+$$

$$+\frac{\lambda_1}{2}\int_s^T \langle f(w_k(t))-f(w_m(t)), w_k(t)-w_m(t)\rangle\, dt\le$$

$$\le(\frac{3}{2}+\lambda_1)E(w_k(s)-w_m(s))+\frac{\lambda_1}{2}\left\langle \widehat{\Sigma}(w_k(s))+\widehat{\Sigma}(w_m(s)),1\right\rangle+$$
$$+\langle F(w_k(s))+F(w_m(s)),1\rangle, \quad 0\le s\le T,$$

where $\widehat{\Sigma}(w)=\int_0^w s\sigma(s)ds$. Integrating the last inequality with respect to s from 0 to T we find

$$\frac{T}{2}E(w_k(T)-w_m(T))+\lambda_1\int_0^T sE(w_k(s)-w_m(s))ds+$$

$$+\int_0^T s\,\langle \sigma(w_k(s))w_{kt}(s)-\sigma(w_m(s))w_{mt}(s), w_{kt}(s)-w_{mt}(s)\rangle\, ds+$$

$$+\frac{\lambda_1}{2}T\left\langle \widehat{\Sigma}(w_k(T))+\widehat{\Sigma}(w_m(T)),1\right\rangle-\frac{\lambda_1}{2}\int_0^T s\,\langle \sigma(w_k(s))w_{kt}(s), w_m(s)\rangle\, ds$$

$$-\frac{\lambda_1}{2}\int_0^T s\,\langle \sigma(w_m(s))w_{mt}(s), w_k(s)\rangle\, ds+T\,\langle F(w_k(T))+F(w_m(T)),1\rangle-$$

$$-\int_0^T s\,\langle f(w_k(s)), w_{mt}(s)\rangle\, ds-\int_0^T s\,\langle f(w_m(s)), w_{kt}(s)\rangle\, ds+$$

$$+\frac{\lambda_1}{2}\int_0^T s\,\langle f(w_k(s))-f(w_m(s)), w_k(s)-w_m(s)\rangle\, dt\le$$

$$\leq (\frac{3}{2}+\lambda_1)\int_0^T E(w_k(s)-w_m(s))ds + \int_0^T \left\langle F(w_k(s)) + \frac{\lambda_1}{2}\widehat{\Sigma}(w_k(s)), 1\right\rangle ds +$$

$$+\int_0^T \left\langle F(w_m(s)) + \frac{\lambda_1}{2}\widehat{\Sigma}(w_m(s)), 1\right\rangle ds, \quad \forall T\geq 0. \tag{3.3}$$

By $(3.1)_1$, it follows that

$$(\frac{3}{2}+\lambda_1)\int_0^T E(w_k(s)-w_m(s))ds \leq c_1 +$$

$$+\frac{\lambda_1}{2}\int_{\frac{3+2\lambda_1}{\lambda_1}}^T sE(w_k(s)-w_m(s))ds, \quad \forall T\geq \frac{3+2\lambda_1}{\lambda_1}. \tag{3.4}$$

Since for every $\varepsilon>0$ the embedding $H^1(\Omega)\subset H^{1-\varepsilon}(\Omega)$ is compact (see for example [12, Theorem 16.1]), applying [18, Corollary 1] to (3.2), we have

$$w_k \to w \text{ strongly in } C([0,T]; H^{1-\varepsilon}(\Omega)).$$

Applying Lemma 2.1 it yields that

$$\begin{cases} \sigma(w_k)\to\sigma(w) \text{ strongly in } C([0,T]; L_{\frac{3}{2}-\varepsilon}(\Omega)), \\ \sigma^{\frac{1}{2}}(w_k)\to\sigma^{\frac{1}{2}}(w) \text{ strongly in } C([0,T]; L_{3-\varepsilon}(\Omega)), \end{cases}$$

for small enough $\varepsilon>0$. The last approximation together with (2.3) and $(3.2)_2$ implies that

$$\begin{cases} \sigma(w_k)w_{kt}\to\sigma(w)w_t \text{ weakly in } L_2([0,T]; L_{\frac{6}{5}}(\Omega)), \\ \sigma^{\frac{1}{2}}(w_k)w_{kt}\to\sigma^{\frac{1}{2}}(w)w_t \text{ weakly in } L_2([0,T]; L_2(\Omega)), \end{cases}$$

by which we obtain

$$\liminf_{m\to\infty}\liminf_{k\to\infty}\int_0^T s\left\langle \sigma(w_k(s))w_{kt}(s)-\sigma(w_m(s))w_{mt}(s), w_{kt}(s)-w_{mt}(s)\right\rangle ds =$$

$$=\liminf_{k\to\infty}\int_0^T s\left\|\sigma^{\frac{1}{2}}(w_k(s))w_{kt}(s)\right\|^2_{L_2(\Omega)} ds + \liminf_{m\to\infty}\int_0^T s\left\|\sigma^{\frac{1}{2}}(w_m(s))w_{mt}(s)\right\|^2_{L_2(\Omega)} ds -$$

$$-2\int_0^T s\left\|\sigma^{\frac{1}{2}}(w(s))w_t(s)\right\|^2_{L_2(\Omega)} ds \geq 0, \tag{3.5}$$

$$\lim_{m\to\infty}\lim_{k\to\infty}\int_0^T s\left\langle\sigma(w_k(s))w_{kt}(s), w_m(s)\right\rangle ds = \int_0^T s\left\langle\sigma(w(s))w_t(s), w(s)\right\rangle ds =$$

$$= T\int_0^T \left\langle\widehat{\Sigma}(w(s)), 1\right\rangle ds - \int_0^T \left\langle\widehat{\Sigma}(w(s)), 1\right\rangle ds \tag{3.6}$$

and

$$\lim_{m\to\infty}\lim_{k\to\infty}\int_0^T s\left\langle \sigma(w_m(s))w_{mt}(s), w_k(s)\right\rangle ds = \int_0^T s\left\langle \sigma(w(s))w_t(s), w(s)\right\rangle ds =$$

$$= T\int_0^T \left\langle \widehat{\Sigma}(w(s)), 1\right\rangle ds - \int_0^T \left\langle \widehat{\Sigma}(w(s)), 1\right\rangle ds \tag{3.7}$$

Also applying Fatou's lemma and using (2.1), (2.2), (2.3), (3.2), we have

$$\begin{cases} \liminf\limits_{k\to\infty}\left\langle \widehat{\Sigma}(w_k(T)), 1\right\rangle \geq \left\langle \widehat{\Sigma}(w(T)), 1\right\rangle, \\ \liminf\limits_{k\to\infty}\left\langle F(w_k(T)), 1\right\rangle \geq \left\langle F(w(T)), 1\right\rangle, \\ \liminf\limits_{k\to\infty}\int_0^T s\left\langle f(w_k(s)), w_k(s)\right\rangle ds \geq \int_0^T s\left\langle f(w(s)), w(s)\right\rangle ds. \end{cases} \tag{3.8}$$

Taking into account (3.4)-(3.8) in (3.3), we obtain

$$\frac{T}{2}\liminf_{m\to\infty}\liminf_{k\to\infty}E(w_k(T)-w_m(T)) + \frac{\lambda_1}{2}\liminf_{m\to\infty}\liminf_{k\to\infty}\int_0^T sE(w_k(s)-w_m(s))ds \leq c_1 +$$

$$+ 2\liminf_{k\to\infty}\int_0^T \left\langle F(w_k(s)) + \frac{\lambda_1}{2}\widehat{\Sigma}(w_k(s)) - F(w(s)) - \frac{\lambda_1}{2}\widehat{\Sigma}(w(s)), 1\right\rangle ds, \tag{3.9}$$

for $T \geq \frac{3+2\lambda_1}{\lambda_1}$. Now let us estimate the right hand side of (3.9). By (2.1), $(3.1)_1$ and (3.2), we find that

$$\int_0^T \left|\left\langle F(w_m(s)) - F(w(s)), 1\right\rangle\right| ds \leq c_2\int_0^T \|w_m(s)-w(s)\|_{H_0^1(\Omega)} ds \leq c_3 + c_4(\varepsilon)\log(T) +$$

$$+\varepsilon\int_1^T s\|w_m(s)-w(s)\|^2_{H_0^1(\Omega)} ds \leq c_3 + c_4(\varepsilon)\log(T) +$$

$$+ \varepsilon\liminf_{k\to\infty}\int_0^T s\|w_m(s)-w_k(s)\|^2_{H_0^1(\Omega)} ds, \quad \forall T\geq 1, \ \ \forall\varepsilon>0. \tag{3.10}$$

By the same way, we have

$$\int_0^T \left|\left\langle \widehat{\Sigma}(w_m(s)) - \widehat{\Sigma}(w(s)), 1\right\rangle\right| ds \leq c_5 + c_6(\varepsilon)\log(T) +$$

$$+ \varepsilon\liminf_{k\to\infty}\int_0^T s\|w_m(s)-w_k(s)\|^2_{H_0^1(\Omega)} ds, \quad \forall T\geq 1, \ \ \forall\varepsilon>0. \tag{3.11}$$

Now, choosing ε small enough, by (3.9)-(3.11), we obtain

$$\liminf_{m\to\infty}\liminf_{k\to\infty}E(w_k(T)-w_m(T)) \leq \frac{c_7(1+\log(T))}{T}, \quad \forall T\geq \max\left\{1, \frac{3+2\lambda_1}{\lambda_1}\right\}.$$

Choosing $T=T_0$ in the last inequality we find

$$\liminf_{n\to\infty}\liminf_{m\to\infty}\|S(t_n)\varphi_n-S(t_m)\varphi_m\|_{\mathcal{H}}\le c_8\sqrt{\frac{(1+\log(T_0))}{T_0}},$$

and passing to the limit as $T_0\to\infty$ we have

$$\liminf_{n\to\infty}\liminf_{m\to\infty}\|S(t_n)\varphi_n-S(t_m)\varphi_m\|_{\mathcal{H}}=0.$$

Similarly one can show that

$$\liminf_{k\to\infty}\liminf_{m\to\infty}\|S(t_{n_k})\varphi_{n_k}-S(t_{n_m})\varphi_{n_m}\|_{\mathcal{H}}=0, \tag{3.12}$$

for every subsequence $\{n_k\}_{k=1}^{\infty}$. Now if the sequence $\{S(t_n)\varphi_n\}_{n=1}^{\infty}$ has no convergent subsequence in $\mathcal{H}$, then there exist $\varepsilon_0>0$ and a subsequence $\{n_k\}_{k=1}^{\infty}$, such that

$$\|S(t_{n_k})\varphi_{n_k}-S(t_{n_m})\varphi_{n_m}\|_{\mathcal{H}}\ge\varepsilon_0,\quad k\neq m.$$

The last inequality contradicts (3.12). □

Now since, by (2.4), the problem (1.1)-(1.3) has a strict Lyapunov function $L(w(t)):=E(w(t))+\langle F(w(t)),1\rangle-\langle g,w(t)\rangle$, according to [4, Corollary 2.29] we have the following theorem:

Theorem 3.1. *Under conditions (2.1)-(2.3), the semigroup $\{S(t)\}_{t\ge0}$ possesses a global attractor $\mathcal{A}_{\mathcal{H}}$ in $\mathcal{H}$.*

4. Existence of the global attractor in $\mathcal{H}_1$

To prove the existence of a global attractor in $\mathcal{H}_1$ we need the following lemmas:

Lemma 4.1. *Let conditions (2.1)-(2.3) hold and B be a bounded subset of $\mathcal{H}_1$. Then*

$$\sup_{t\ge0}\sup_{\varphi\in B}\|S(t)\varphi\|_{\mathcal{H}_1}<\infty. \tag{4.1}$$

Proof. We use the formal estimates which can be justified by Galerkin's approximations. Multiplying both sides of (1.1) by $-\Delta w_t$ and integrating over Ω, we obtain

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\|\nabla w_t(t)\|^2_{L_2(\Omega)}+\frac{1}{2}\|\Delta w(t)\|^2_{L_2(\Omega)}+\langle g,\Delta w(t)\rangle\right)+\\+\frac{1}{2}\|\Delta w_t(t)\|^2_{L_2(\Omega)}\le\|f(w(t))\|^2_{L_2(\Omega)}+\\+\|\sigma(w(t))w_t(t)\|^2_{L_2(\Omega)},\quad\forall t\ge0.\end{aligned} \tag{4.2}$$

By (2.1) and (2.3), we have

$$\begin{aligned}\|f(w(t))\|^2_{L_2(\Omega)}+\|\sigma(w(t))w_t(t)\|^2_{L_2(\Omega)}\le c_1\left(1+\|w(t)\|^{10}_{L_{10}(\Omega)}+\|w_t(t)\|^2_{L_2(\Omega)}\right)+\\+c_2\|w(t)\|^8_{L_{10}(\Omega)}\|w_t(t)\|^2_{L_{10}(\Omega)},\ \forall t\ge0.\end{aligned} \tag{4.3}$$

On the other hand, by the embedding and interpolation theorems, we find

$$\|\varphi\|_{L_{10}(\Omega)}\le c_2\|\varphi\|_{H^{\frac{6}{5}}(\Omega)}\le c_3\|\varphi\|^{\frac{1}{5}}_{H^2(\Omega)}\|\varphi\|^{\frac{4}{5}}_{H^1(\Omega)},\quad\forall\varphi\in H^2(\Omega). \tag{4.4}$$

Taking into account (2.4), (4.3) and (4.4) in (4.2) and applying Gronwall's lemma, we obtain

$$\|(w(t),w_t(t))\|_{\mathcal{H}_1}\le C(t,r)(1+\|(w_0,w_1)\|_{\mathcal{H}_1}),\quad\forall t\ge0, \tag{4.5}$$

where $C : R_+ \times R_+ \to R_+$ is a nondecreasing function with respect to each variable and $r = \sup\limits_{\varphi \in B} \|\varphi\|_{\mathcal{H}}$. Since the embedding $\mathcal{H}_1 \subset \mathcal{H}$ is compact, by (4.5), it follows that the set $\underset{0 \leq t \leq T}{\cup} S(t)B$ is a relatively compact subset of $\mathcal{H}$, for every $T > 0$. This together with Lemma 3.1 implies the relative compactness of $\underset{t \geq 0}{\cup} S(t)B$ in $\mathcal{H}$. Now using this fact let us estimate $\|w(t)\|_{L_{10}(\Omega)}$:

$$\|w(t)\|_{L_{10}(\Omega)}^{10} \leq m^{10} mes(\Omega) + \int\limits_{\{x : x \in \Omega,\ |w(t,x)| > m\}} |w(t,x)|^{10} \, dx \leq$$

$$\leq m^{10} mes(\Omega) + \left(\int\limits_{\{x : x \in \Omega,\ |w(t,x)| > m\}} |w(t,x)|^6 \, dx \right)^{\frac{1}{3}} \|w(t)\|_{L_{12}(\Omega)}^8 \leq$$

$$\leq m^{10} mes(\Omega) + c_4 \left(\int\limits_{\{x : x \in \Omega,\ |w(t,x)| > m\}} |w(t,x)|^6 \, dx \right)^{\frac{1}{3}} \|w(t)\|_{H^2(\Omega)}^2 \|w(t)\|_{H^1(\Omega)}^6 .$$

So for any $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$\|w(t)\|_{L_{10}(\Omega)} \leq \varepsilon \|\Delta w(t)\|_{L_2(\Omega)}^{\frac{1}{5}} + c_\varepsilon, \ \forall t \geq 0,$$

which together with (4.2)-(4.4) yields

$$\frac{d}{dt} \left(\frac{1}{2} \|\nabla w_t(t)\|_{L_2(\Omega)}^2 + \frac{1}{2} \|\Delta w(t)\|_{L_2(\Omega)}^2 + \langle g, \Delta w(t) \rangle \right) + \frac{1}{4} \|\Delta w_t(t)\|_{L_2(\Omega)}^2 \leq$$

$$\leq c_5 \|\nabla w_t(t)\|_{L_2(\Omega)}^2 \|\Delta w(t)\|_{L_2(\Omega)}^2 + \varepsilon \|\Delta w(t)\|_{L_2(\Omega)}^2 + \widetilde{c}_\varepsilon + c_5, \quad \forall t \geq 0.$$

Now multiplying both sides of (1.1) by $-\mu \Delta w$ ($\mu \in (0,1)$) and integrating over Ω, we obtain

$$\frac{d}{dt} \left(\frac{1}{2} \mu \|\Delta w(t)\|_{L_2(\Omega)}^2 + \mu \langle \nabla w_t(t), \nabla w(t) \rangle \right) + \mu \|\Delta w(t)\|_{L_2(\Omega)}^2 \leq$$

$$\leq \mu \|g\|_{L_2(\Omega)} \|\Delta w(t)\|_{L_2(\Omega)} + \mu \|\nabla w_t(t)\|_{L_2(\Omega)}^2 + \mu \|\sigma(w(t)) w_t(t)\|_{L_2(\Omega)} \|\Delta w(t)\|_{L_2(\Omega)}$$
$$+ \mu \|f(w(t))\|_{L_2(\Omega)} \|\Delta w(t)\|_{L_2(\Omega)}, \quad \forall t \geq 0.$$

Taking into account the relative compactness of $\underset{t \geq 0}{\cup} S(t)B$, similar to the argument done above, we can say that for any $\varepsilon > 0$ there exists $\widehat{c}_\varepsilon > 0$ such that

$$\|f(w(t))\|_{L_2(\Omega)}^2 + \|\sigma(w(t)) w_t(t)\|_{L_2(\Omega)}^2 \leq \varepsilon \left(\|\Delta w(t)\|_{L_2(\Omega)}^2 + \|\Delta w_t(t)\|_{L_2(\Omega)}^2 \right) +$$

$$+ \widehat{c}_\varepsilon \|\Delta w(t)\|_{L_2(\Omega)}^2 \|\nabla w_t(t)\|_{L_2(\Omega)}^2 + \widehat{c}_\varepsilon, \ \forall t \geq 0.$$

By the last three inequalities we have

$$\frac{d}{dt} \left(\frac{1}{2} \|\nabla w_t(t)\|_{L_2(\Omega)}^2 + \frac{1}{2}(1+\mu) \|\Delta w(t)\|_{L_2(\Omega)}^2 + \mu \langle \nabla w_t(t), \nabla w(t) \rangle + \langle g, \Delta w(t) \rangle \right)$$

$$+ (\frac{1}{4} - \mu c_6 - \varepsilon) \|\Delta w_t(t)\|_{L_2(\Omega)}^2 + (\frac{1}{4}\mu - 2\varepsilon) \|\Delta w(t)\|_{L_2(\Omega)}^2 \leq$$

$$\leq (c_5 + \widehat{c}_\varepsilon) \|\Delta w(t)\|_{L_2(\Omega)}^2 \|\nabla w_t(t)\|_{L_2(\Omega)}^2 + c_6 + \widehat{c}_\varepsilon + \widetilde{c}_\varepsilon, \ \forall t \geq 0.$$

Choosing μ small enough and $\varepsilon \in (0, \frac{1}{8}\mu)$, we obtain

$$\frac{d}{dt}\Phi(t) + c_7\Phi(t) \le c_8 \|\nabla w_t(t)\|^2_{L_2(\Omega)} \Phi(t) + c_8(1 + \|\nabla w_t(t)\|^2_{L_2(\Omega)}),\ \forall t \ge 0,$$

where $\Phi(t) = \frac{1}{2}\|\nabla w_t(t)\|^2_{L_2(\Omega)} + \frac{1}{2}(1+\mu)\|\Delta w(t)\|^2_{L_2(\Omega)} + \mu\langle\nabla w_t(t), \nabla w(t)\rangle +$ $+ \langle g, \Delta w(t)\rangle$. Multiplying both sides of the last inequality by $e^{\int_0^t (c_7 - c_8\|\nabla w_t(\tau)\|^2_{L_2(\Omega)})d\tau}$, integrating over $[0,T]$ and multiplying both sides of obtained inequality by $e^{-\int_0^T [c_7 - c_8\|\nabla w_t(t)\|^2_{L_2(\Omega)}]dt}$, we find

$$\Phi(T) \le \Phi(0)e^{-\int_0^T (c_7 - c_8\|\nabla w_t(t)\|^2_{L_2(\Omega)})dt} +$$

$$+c_8\int_0^T (1 + \|\nabla w_t(t)\|^2_{L_2(\Omega)})e^{-\int_t^T (c_7 - c_8\|\nabla w_t(\tau)\|^2_{L_2(\Omega)})d\tau} dt,\ \forall T \ge 0,$$

which together with (2.4) yields (4.1). □

Lemma 4.2. *Let conditions (2.1)-(2.3) hold and B be a bounded subset of $\mathcal{H}_1$. Then every sequence of the form $\{S(t_n)\varphi_n\}_{n=1}^\infty$, $\{\varphi_n\}_{n=1}^\infty \subset B$, $t_n \to \infty$, has a convergent subsequence in $\mathcal{H}_1$.*

Proof. Let us decompose $\{S(t)\}_{t\ge0}$ as $S(t) = U(t) + C(t)$, where $U(t)$ is a linear semigroup generated by the problem

$$\begin{cases} u_{tt} - \Delta u_t - \Delta u = 0, & \text{in} \quad (0,\infty)\times\Omega, \\ u = 0, & \text{on} \quad (0,\infty)\times\partial\Omega, \\ u(0,\cdot) = w_0\ , \quad u_t(0,\cdot) = w_1, & \text{in} \quad \Omega, \end{cases} \tag{4.6}$$

$C(t)$ is a solution operator of

$$\begin{cases} v_{tt} - \Delta v_t - \Delta v = g(x) - f(w) - \sigma(w)w_t, & \text{in} \quad (0,\infty)\times\Omega, \\ v = 0, & \text{on} \quad (0,\infty)\times\partial\Omega, \\ v_k(0,\cdot) = 0\ , \quad v_t(0,\cdot) = 0, & \text{in} \quad \Omega \end{cases} \tag{4.7}$$

(i.e. $(u(t), u_t(t)) = U(t)(w_0, w_1)$ and $(v(t), v_t(t)) = C(t)(w_0, w_1)$) and $(w(t), w_t(t))=$ $S(t)(w_0, w_1)$. Multiplying $(4.6)_1$ by $(u_t - \frac{1}{2}\Delta u - \mu\Delta u_t - \nu t\Delta u_t)$ and integrating over Ω, we obtain

$$\frac{d}{dt}\left(E(u(t)) + \frac{1}{4}\|\Delta u(t)\|^2_{L_2(\Omega)} - \frac{1}{2}\langle u_t, \Delta u\rangle + \frac{1}{2}(\mu + \nu t)\|\nabla u_t(t)\|^2_{L_2(\Omega)} + \right.$$

$$\left. + \frac{1}{2}(\mu + \nu t)\|\Delta u(t)\|^2_{L_2(\Omega)}\right) + \frac{1}{2}(1-\nu)\|\nabla u_t(t)\|^2_{L_2(\Omega)} + \frac{1}{2}(1-\nu)\|\Delta u(t)\|^2_{L_2(\Omega)} +$$

$$+(\mu + \nu t)\|\Delta u_t(t)\|^2_{L_2(\Omega)} = 0, \qquad \forall t \ge 0.$$

Choosing $(\mu, \nu) = (1, 0)$ and $(\mu, \nu) = (0, 1)$ in the last equality, we find

$$\|U(t)\|_{\mathcal{L}(\mathcal{H}_1, \mathcal{H}_1)} \le Me^{-\omega t}, \qquad \forall t \ge 0, \tag{4.8}$$

and

$$\|U(t)\|_{\mathcal{L}((H^2(\Omega)\cap H_0^1(\Omega))\times L_2(\Omega), \mathcal{H}_1)} \le \frac{M}{\sqrt{t}}, \quad \forall t > 0, \tag{4.9}$$

respectively, where $M > 0$ and $\omega > 0$. Also applying Duhamel's principle to (4.7), we have

$$C(t)(w_0, w_1) = \int_0^t U(t-s)(0, \Phi_{(w_0,w_1)}(s))ds, \tag{4.10}$$

where $\Phi_{(w_0,w_1)}(s) = g - f(w(s)) - \sigma(w(s))w_t(s)$. By Lemma 4.1 and equation (1.1), it follows that the set of functions $\left\{\Phi_{(w_0,w_1)}(s) : (w_0, w_1) \in B \ \right\}$ is precompact in $C([0,t]; L_2(\Omega))$. So, from (4.9) and (4.10) we obtain that the operator $C(t) : \mathcal{H}_1 \to \mathcal{H}_1$, $t \geq 0$, is compact. Since

$$S(t_n)\varphi_n = U(T)S(t_n - T)\varphi_n + C(T)S(t_n - T)\varphi_n$$

for $t_n \geq T$, by (4.1), (4.8) and the compactness of $C(t)$, we obtain that the sequence $\{S(t_n)\varphi_n\}_{n=1}^{\infty}$ has a finite ε-net in $\mathcal{H}$, for every $\varepsilon > 0$. This completes the proof. $\square$

Now by Lemma 4.2, similar to Theorem 3.1, we obtain the following theorem:

Theorem 4.1. *Under conditions (2.1)-(2.3), the semigroup $\{S(t)\}_{t\geq 0}$ possesses a global attractor $\mathcal{A}_{\mathcal{H}_1}$ in $\mathcal{H}_1$.*

5. Regularity of the $\mathcal{A}_{\mathcal{H}}$

To prove the regularity of $\mathcal{A}_{\mathcal{H}}$ we will use the method used in [9] and [10]. Since $\mathcal{A}_{\mathcal{H}}$ is invariant, by [1, p. 159], for every $(w_0, w_1) \in \mathcal{A}_{\mathcal{H}}$ there exists an invariant trajectory $\gamma = \{W(t) = (w(t), w_t(t)),\, t \in R\} \subset \mathcal{A}_{\mathcal{H}}$ such that $W(0) = (w_0, w_1)$. By an invariant trajectory we mean a curve $\gamma = \{W(t),\, t \in R\}$ such that $S(t)W(\tau) = W(t+\tau)$ for $t \geq 0$ and $\tau \in R$ (see [1, p. 157]). Let us decompose $w(t)$ as $w(t) = u_k(t,s) + v_k(t,s)$, where

$$\begin{cases} v_{ktt} - \Delta v_{kt} + \sigma_k(w)v_{kt} - \Delta v_k + f_k(w) = g(x), & \text{in } \ (s,\infty)\times\Omega, \\ v_k = 0, & \text{on } \ (s,\infty)\times\partial\Omega, \\ v_k(s,s,\cdot) = 0\ , \qquad v_{kt}(s,s,\cdot) = 0, & \text{in } \ \Omega \end{cases}, \tag{5.1}$$

$$\begin{cases} u_{ktt} - \Delta u_{kt} + \sigma(w)w_t - \sigma_k(w)v_{kt} - \Delta u_k = \\ = f_k(w) - f(w), & \text{in } \ (s,\infty)\times\Omega, \\ u_k = 0, & \text{on } \ (s,\infty)\times\partial\Omega, \\ u_k(s,s,\cdot) = w(s,\cdot)\ , \qquad u_{kt}(s,s,\cdot) = w_t(s,\cdot), & \text{in } \ \Omega \end{cases}, \tag{5.2}$$

$$f_k(s) = \begin{cases} f(k), & s > k, \\ f(s), & |s| \leq k, \\ f(-k), & s < -k \end{cases}, \quad \sigma_k(s) = \begin{cases} \sigma(k), & s > k, \\ \sigma(s), & |s| \leq k, \\ \sigma(-k), & s < -k \end{cases} \quad \text{and } k \in \mathbb{N}.$$

Now let us prove the following lemmas:

Lemma 5.1. *Assume that conditions (2.1)-(2.3) are satisfied. Then $(v_k(t,s), v_{kt}(t,s)) \in \mathcal{H}_1$ and for any $k \in \mathbb{N}$ there exists $T_k < 0$ such that*

$$\|v_{kt}(t,s)\|_{H^1(\Omega)} + \|v_k(t,s)\|_{H^2(\Omega)} \leq r_0 k^{\frac{128}{65}}, \ \forall s \leq t \leq T_k, \tag{5.3}$$

where the positive constant r_0 is independent of k and (w_0, w_1).

Proof. Multiplying both sides of $(5.1)_1$ by $v_{kt} + \mu v_k$ ($\mu \in (0,1)$) and integrating over Ω, we obtain

$$\frac{d}{dt}\Big(E(v_k(t,s)) + \frac{\mu}{2}\|\nabla v_k(t,s)\|^2_{L_2(\Omega)} + \mu\left\langle v_{kt}(t,s), v_k(t,s)\right\rangle\Big) +$$

$$+\frac{1}{2}\left\|\nabla v_{kt}(t,s)\right\|^2_{L_2(\Omega)}-\mu\left\|v_{kt}(t,s)\right\|^2_{L_2(\Omega)}+(\mu-c_1\mu^2)\left\|\nabla v_k(t,s)\right\|^2_{L_2(\Omega)}\le c_2,\ \forall t\ge s.$$

Choosing μ small enough in the last inequality, we find

$$\left\|v_{kt}(t,s)\right\|_{L_2(\Omega)}+\left\|v_k(t,s)\right\|_{H_0^1(\Omega)}\le c_3,\quad \forall t\ge s. \tag{5.4}$$

Multiplying both sides of $(5.1)_1$ by v_{kt}, integrating over $(\tau_1,\tau_2)\times\Omega$ and taking into account (5.4), we have

$$\int_{\tau_1}^{\tau_2}\left\|\nabla v_{kt}(t,s)\right\|^2_{L_2(\Omega)}dt\le c_4+\int_{\tau_1}^{\tau_2}\left|\left\langle f_k'(w(t))w_t(t),v_k(t,s)\right\rangle\right|dt\le c_4+$$

$$+c_5\int_{\tau_1}^{\tau_2}\left\|\nabla w_t(t)\right\|_{L_2(\Omega)}dt,\quad \forall\tau_2\ge\tau_1\ge s. \tag{5.5}$$

On the other hand, by (2.4), we have

$$\int_{-\infty}^{\infty}\left\|\nabla w_t(t)\right\|^2_{L_2(\Omega)}dt<\infty, \tag{5.6}$$

which together with (5.5) yields

$$\int_{\tau_1}^{\tau_2}\left\|\nabla v_{kt}(t,s)\right\|^2_{L_2(\Omega)}dt\le c_6(1+(\tau_2-\tau_1)^{\frac{1}{2}}),\quad \forall\tau_2\ge\tau_1\ge s. \tag{5.7}$$

Multiplying both sides of $(5.1)_1$ by $-\Delta v_{kt}-\mu\Delta v_k$ ($\mu\in(0,1)$), integrating over Ω and taking into account (5.4), we have

$$\frac{d}{dt}\left(\frac{1}{2}\left\|\nabla v_{kt}(t,s)\right\|^2_{L_2(\Omega)}+\frac{1}{2}\left\|\Delta v_k(t,s)\right\|^2_{L_2(\Omega)}+\mu\left\langle\nabla v_{kt}(t,s),\nabla v_k(t,s)\right\rangle\right)+$$

$$+(\frac{1}{2}-c_7\mu)\left\|\Delta v_{kt}(t,s)\right\|^2_{L_2(\Omega)}+(\mu-\mu^2)\left\|\Delta v_k(t,s)\right\|^2_{L_2(\Omega)}\le c_7+$$

$$+c_7\left\|\sigma_k(w(t))v_{kt}(t,s)\right\|^2_{L_2(\Omega)}+c_7\left\|f_k(w(t))\right\|^2_{L_2(\Omega)},\ \forall t\ge s. \tag{5.8}$$

Now let us estimate the last two terms on the right side of (5.8). By (4.4) and (5.4), we find

$$\left\|\sigma_k(w(t))v_{kt}(t,s)\right\|^2_{L_2(\Omega)}\le\left\|\sigma_k(w(t))\right\|^2_{L_{\frac{5}{2}}(\Omega)}\left\|v_{kt}(t,s)\right\|^2_{L_{10}(\Omega)}\le$$

$$\le c_8\left\|\sigma_k(w(t))\right\|^2_{L_{\frac{5}{2}}(\Omega)}\left\|v_{kt}(t,s)\right\|^{\frac{2}{5}}_{H^2(\Omega)}\left\|v_{kt}(t,s)\right\|^{\frac{8}{5}}_{H^1(\Omega)}\le$$

$$\le c_9\left\|\sigma_k(w(t))\right\|^4_{L_{\frac{5}{2}}(\Omega)}+c_9\left\|\Delta v_{kt}(t,s)\right\|^2_{L_2(\Omega)}\left\|\nabla v_{kt}(t,s)\right\|^2_{L_2(\Omega)}+$$

$$+\frac{1}{3c_7}\left\|\Delta v_{kt}(t,s)\right\|^2_{L_2(\Omega)},\ \forall t\ge s. \tag{5.9}$$

Also by the definitions of $\sigma_k(\cdot)$ and $f_k(\cdot)$, we have

$$\left\|\sigma_k(w(t))\right\|^{\frac{5}{2}}_{L_{\frac{5}{2}}(\Omega)}=\int_\Omega\left|\sigma_k(w(t,x))\right|^{\frac{5}{2}}dx\le$$

$$\le\int_{\{x:x\in\Omega,\ |w(t,x)|\le 2m\}}\left|\sigma_k(w(t,x))\right|^{\frac{5}{2}}dx+\int_{\{x:x\in\Omega,\ |w(t,x)|>2m\}}\left|\sigma_k(w(t,x))\right|^{\frac{5}{2}}dx\le$$

$$\leq c_{10}m^4 \int\limits_{\{x:x\in\Omega,\ |w(t,x)|\leq 2m\}} (1+|w(t,x)|^6)dx+$$

$$+c_{10}k^4 \int\limits_{\{x:x\in\Omega,\ |w(t,x)|>2m\}} |w(t,x)|^6\,dx \leq c_{11}m^4+$$

$$+\,c_{10}k^4 \int\limits_{\{x:x\in\Omega,\ |w(t,x)|>2m\}} |w(t,x)|^6\,dx, \quad \forall k\in\mathbb{N}\ \forall m\geq 1 \text{ and } \forall t\in R. \tag{5.10}$$

$$\|f_k(w(t))\|^2_{L_2(\Omega)} = \int\limits_{\Omega} |f_k(w(t,x))|^2\,dx \leq$$

$$\leq c_{12}m^4 \int\limits_{\{x:x\in\Omega,\ |w(t,x)|\leq 2m\}} (1+|w(t,x)|^6)dx+$$

$$+c_{12}k^4 \int\limits_{\{x:x\in\Omega,\ |w(t,x)|>2m\}} |w(t,x)|^6\,dx \leq c_{13}m^4+$$

$$c_{12}k^4 \int\limits_{\{x:x\in\Omega,\ |w(t,x)|>2m\}} |w(t,x)|^6\,dx, \quad \forall k\in\mathbb{N}\ \forall m\geq 1 \text{ and } \forall t\in R. \tag{5.11}$$

Now denote $w^{(m)}(t,x)=\begin{cases} w(t,x)-m, & w(t,x)>m \\ 0, & |w(t,x)|\leq m \\ w(t,x)+m, & w(t,x)<-m \end{cases}$. Since,

$$|w(t,x)| < 2\left|w^{(m)}(t,x)\right|, \qquad \forall (t,x)\in\{(t,x)\in R\times\Omega,\ |w(t,x)|>2m\},$$

we have

$$\int\limits_{\{x:x\in\Omega,\ |w(t,x)|>2m\}} |w(t,x)|^6\,dx \leq 2^6 \int\limits_{\{x:x\in\Omega,\ |w(t,x)|>2m\}} |w^m(t,x)|^6\,dx \leq$$

$$\leq 2^6 \int\limits_{\Omega} |w^m(t,x)|^6\,dx \leq c_{14}\left\|\nabla w^{(m)}(t)\right\|^2_{L_2(\Omega)}, \quad \forall t\in R. \tag{5.12}$$

So, by (5.8)-(5.12), it follows that

$$\frac{d}{dt}\left(\frac{1}{2}\|\nabla v_{kt}(t,s)\|^2_{L_2(\Omega)} + \frac{1}{2}\|\Delta v_k(t,s)\|^2_{L_2(\Omega)} + \mu\langle\nabla v_{kt}(t,s),\nabla v_k(t,s)\rangle\right)+$$

$$+(\frac{1}{6}-c_7\mu)\|\Delta v_{kt}(t,s)\|^2_{L_2(\Omega)} + (\mu-\mu^2)\|\Delta v_k(t,s)\|^2_{L_2(\Omega)} \leq c_{15}m^{\frac{32}{5}}+$$

$$+c_{15}\|\Delta v_{kt}(t,s)\|^2_{L_2(\Omega)}\|\nabla v_{kt}(t,s)\|^2_{L_2(\Omega)}+$$

$$+\,c_{15}k^{\frac{32}{5}}\left\|\nabla w^{(m)}(t)\right\|^2_{L_2(\Omega)}, \quad \forall k\in\mathbb{N}\ \forall m\geq 1 \text{ and } \forall t\geq s. \tag{5.13}$$

On the other hand, testing (1.1) by $w^{(m)}$, we obtain

$$\frac{d}{dt}\left\langle w_t(t),w^{(m)}(t)\right\rangle + \left\|\nabla w^{(m)}(t)\right\|^2_{L_2(\Omega)} - \left\|w_t^{(m)}(t)\right\|^2_{L_2(\Omega)} + \left\langle\nabla w_t(t),\nabla w^{(m)}(t)\right\rangle =$$

$$= \left\langle g,w^{(m)}(t)\right\rangle - \left\langle\sigma(w(t))w_t(t),w^{(m)}(t)\right\rangle - \left\langle f(w(t)),w^{(m)}(t)\right\rangle,\ \forall t\in R. \tag{5.14}$$

Let us estimate each term on the right hand side of (5.14). By the definition of $w^{(m)}$, we have

$$\left\langle g, w^{(m)}(t)\right\rangle \leq \left(\int\limits_{\{x: x\in\Omega,\ |w(t,x)|>m\}} |g(x)|^{\frac{6}{5}}\, dx \right)^{\frac{5}{6}} \left\| w^{(m)}(t)\right\|_{L_6(\Omega)} \leq$$

$$\leq \frac{c_{16}}{m^2}\left\|\nabla w^{(m)}(t)\right\|_{L_2(\Omega)}, \quad \forall t\in R.$$

By (2.3), it follows that

$$\left|\left\langle \sigma(w(t))w_t(t), w^{(m)}(t)\right\rangle\right| \leq c_{17}\left\|\nabla w_t(t)\right\|_{L_2(\Omega)}\left\|\nabla w^{(m)}(t)\right\|_{L_2(\Omega)}, \quad \forall t\in R.$$

Also by (2.3), we obtain

$$\left\langle f(w(t)), w^{(m)}(t)\right\rangle > -\lambda_1 \left\langle w(t), w^{(m)}(t)\right\rangle \geq$$

$$\geq -\lambda_1 \left(\int\limits_{\{x: x\in\Omega,\ |w(t,x)|>m\}} |w(t,x)|^{\frac{6}{5}}\, dx \right)^{\frac{5}{6}} \left\| w^{(m)}(t)\right\|_{L_6(\Omega)} \geq$$

$$\geq -\frac{c_{18}}{m^4}\left\|\nabla w^{(m)}(t)\right\|_{L_2(\Omega)}, \quad \forall t\in R,$$

for large enough m. Taking into account the last three inequalities in (5.14), we have

$$\frac{d}{dt}\left\langle w_t(t), w^{(m)}(t)\right\rangle + c_{19}\left\|\nabla w^{(m)}(t)\right\|^2_{L_2(\Omega)} \leq$$

$$\leq c_{20}\left\|\nabla w_t(t)\right\|^2_{L_2(\Omega)} + \frac{c_{20}}{m^4}, \quad \forall t\in R. \tag{5.15}$$

for large enough m. Now multiplying (5.15) by $\frac{c_{15}}{c_{19}}k^{\frac{32}{5}}$, adding to (5.13) and then choosing $m = k^{\frac{8}{13}}$, we get

$$\frac{d}{dt}\Lambda_{k,s}(t) + \widehat{c}_1\Lambda_{k,s}(t) \leq \widehat{c}_2\Lambda_{k,s}(t)\left\|\nabla v_{kt}(t,s)\right\|^2_{L_2(\Omega)} +$$

$$+\widehat{c}_2 k^{\frac{256}{65}} + \widehat{c}_2 k^{\frac{32}{5}}\left\|\nabla w_t(t)\right\|^2_{L_2(\Omega)} + \widehat{c}_2 k^{\frac{64}{5}}\left|\left\langle w_t(t), w^{(k^{\frac{13}{8}})}(t)\right\rangle\right|^2, \quad \forall t\geq s,$$

for large enough k and small enough μ, where $\widehat{c}_1$ and $\widehat{c}_2$ are positive constants and $\Lambda_{k,s}(t) := \frac{1}{2}\left\|\nabla v_{kt}(t,s)\right\|^2_{L_2(\Omega)} + \frac{1}{2}\left\|\Delta v_k(t,s)\right\|^2_{L_2(\Omega)} + \mu\left\langle \nabla v_{kt}(t,s), \nabla v_k(t,s)\right\rangle + \frac{c_{15}}{c_{16}}k^{\frac{32}{5}}\left\langle w_t(t), w^{(k^{\frac{8}{13}})}(t)\right\rangle$. Since

$$\left|\left\langle w_t(t), w^{(k^{\frac{8}{13}})}(t)\right\rangle\right| \leq \left\|w_t(t)\right\|_{L_6(\Omega)} \left(\int\limits_{\left\{x: x\in\Omega,\ |w(t,x)|>k^{\frac{8}{13}}\right\}} |w(t,x)|^{\frac{6}{5}}\, dx \right)^{\frac{5}{6}} \leq$$

$$\leq \frac{\widehat{c}_3}{k^{\frac{32}{13}}}\left\|\nabla w_t(t)\right\|_{L_2(\Omega)}, \quad \forall t\in R,$$

by the last differential inequality, we obtain

$$\frac{d}{dt}\Lambda_{k,s}(t) + \widehat{c}_1\Lambda_{k,s}(t) \leq \widehat{c}_2\Lambda_{k,s}(t)\left\|\nabla v_{kt}(t,s)\right\|^2_{L_2(\Omega)} +$$

$$+\widehat{c}_4 k^{\frac{256}{65}} + \widehat{c}_4 k^8 \left\| \nabla w_t(t) \right\|^2_{L_2(\Omega)}, \quad \forall t \geq s.$$

Multiplying both sides of the above inequality by $e^{\int_s^t \left[\widehat{c}_1 - \widehat{c}_2 \|\nabla v_{kt}(\tau,s)\|^2_{L_2(\Omega)}\right] d\tau}$, integrating over $[s,T]$, multiplying both sides of the obtained inequality by $e^{-\int_s^T \left[\widehat{c}_1 - \widehat{c}_2 \|\nabla v_{kt}(t,s)\|^2_{L_2(\Omega)}\right] dt}$ and taking into account (5.7), we find

$$\Lambda_{k,s}(T) \leq \widehat{c}_5 k^{\frac{32}{5}} \left| \left\langle w_t(s), w^{(m)}(s) \right\rangle \right| + \widehat{c}_5 k^{\frac{256}{65}} + \\ + \widehat{c}_5 k^8 \int_s^T \left\| \nabla w_t(t) \right\|^2_{L_2(\Omega)} dt, \quad \forall T \geq s, \tag{5.16}$$

for large enough k and small enough μ. On the other hand, since $\mathcal{A}_{\mathcal{H}}$ is compact subset of $\mathcal{H}$ and problem (1.1)-(1.3) admits a strict Lyapunov function, we have

$$w_t(t) \to 0 \text{ strongly in } L_2(\Omega) \text{ as } t \to -\infty \tag{5.17}$$

Thus, by (5.6) and (5.17), for any $k \in \mathbb{N}$ there exists $T_k = T_k(\gamma) < 0$ such that

$$\widehat{c}_5 k^{\frac{32}{5}} \left| \left\langle w_t(T), w^{(m)}(T) \right\rangle \right| + \widehat{c}_5 k^8 \int_{-\infty}^T \left\| \nabla w_t(t) \right\|^2_{L_2(\Omega)} dt \leq 1, \quad \forall T \leq T_k,$$

which together with (5.16) yields (5.3). □

Lemma 5.2. *Assume that conditions (2.1)-(2.3) are satisfied. Then there exists* $k_0 \in \mathbb{N}$ *such that*

$$\lim_{s \to -\infty} \left(\left\| u_{k_0 t}(t,s) \right\|_{L_2(\Omega)} + \left\| u_{k_0}(t,s) \right\|_{H^1(\Omega)} \right) = 0, \quad \forall t \leq T_{k_0} \tag{5.18}$$

Proof. Multiplying both sides of $(5.2)_1$ by $u_{kt} + \mu u_k$ $(\mu \in (0,1))$ and integrating over Ω, we obtain

$$\frac{d}{dt}\left(E(u_k(t,s)) + \frac{\mu}{2} \left\| \nabla u_k(t,s) \right\|^2_{L_2(\Omega)} + \mu \left\langle u_{kt}(t,s), u_k(t,s) \right\rangle \right) + \\ + \left\| \nabla u_{kt}(t,s) \right\|^2_{L_2(\Omega)} + \mu \left\| \nabla u_k(t,s) \right\|^2_{L_2(\Omega)} - \mu \left\| u_{kt}(t,s) \right\|^2_{L_2(\Omega)} \leq \\ \leq \left\| \sigma(w(t)) - \sigma_k(w(t)) \right\|_{L_{\frac{3}{2}}(\Omega)} \left\| v_{kt}(t,s) \right\|_{L_6(\Omega)} \left\| u_{kt}(t,s) \right\|_{L_6(\Omega)} + \\ + \mu \left\| \sigma(w(t)) \right\|_{L_{\frac{3}{2}}(\Omega)} \left\| u_{kt}(t,s) \right\|_{L_6(\Omega)} \left\| u_k(t,s) \right\|_{L_6(\Omega)} + \\ + \mu \left\| \sigma(w(t)) - \sigma_k(w(t)) \right\|_{L_{\frac{3}{2}}(\Omega)} \left\| v_{kt}(t,s) \right\|_{L_6(\Omega)} \left\| u_k(t,s) \right\|_{L_6(\Omega)} + \\ + \left\| f(w(t)) - f_k(w(t)) \right\|_{L_{\frac{6}{5}}(\Omega)} \left\| u_{kt}(t,s) \right\|_{L_6(\Omega)} + \\ + \mu \left\| f(w(t)) - f_k(w(t)) \right\|_{L_{\frac{6}{5}}(\Omega)} \left\| u_k(t,s) \right\|_{L_6(\Omega)}, \quad \forall t \geq s. \tag{5.19}$$

Taking into account (2.4) in (5.19) and choosing μ small enough, we find

$$\frac{d}{dt}\left(E(u_k(t,s)) + \frac{\mu}{2} \left\| \nabla u_k(t,s) \right\|^2_{L_2(\Omega)} + \mu \left\langle u_{kt}(t,s), u_k(t,s) \right\rangle \right) + \\ + c_1 \left(E(u_k(t,s)) + \frac{\mu}{2} \left\| \nabla u_k(t,s) \right\|^2_{L_2(\Omega)} + \mu \left\langle u_{kt}(t,s), u_k(t,s) \right\rangle \right) \leq \\ \leq c_2 \left\| \sigma(w(t)) - \sigma_k(w(t)) \right\|^2_{L_{\frac{3}{2}}(\Omega)} \left\| v_{kt}(t,s) \right\|^2_{L_6(\Omega)} + \\ + c_2 \left\| f(w(t)) - f_k(w(t)) \right\|^2_{L_{\frac{6}{5}}(\Omega)}, \quad s \leq t \leq T_k, \tag{5.20}$$

where c_1 and c_2 are positive constants. Now let us estimate the terms on the right side of (5.20). Since $H^{\frac{3}{2}+\varepsilon}(\Omega) \subset C(\overline{\Omega})$ and

$$\|\varphi\|_{H^{\frac{3}{2}+\varepsilon}(\Omega)} \leq c_3(\varepsilon) \|\varphi\|_{H^1(\Omega)}^{\frac{1}{2}-\varepsilon} \|\varphi\|_{H^2(\Omega)}^{\frac{1}{2}+\varepsilon}, \ \forall \varphi \in H^2(\Omega), \ \forall \varepsilon \in (0, \frac{1}{2}],$$

from (5.3) and (5.4) it follows that

$$\|v_k(t,s)\|_{C(\overline{\Omega})} \leq \frac{1}{2}k, \quad s \leq t \leq T_k,$$

for large enough k. The last inequality together with (2.1)-(2.4) yields that

$$\begin{aligned}
\|\sigma(w(t)) - \sigma_k(w(t))\|_{L_{\frac{3}{2}}(\Omega)}^{\frac{3}{2}} &\leq c_4 \int\limits_{\{x:x\in\Omega, |w(t,x)|>k\}} |w(t,x)|^6 \, dx \leq \\
&\leq c_5 \left(\int\limits_{\{x:x\in\Omega, |w(t,x)|>k\}} |w(t,x)|^6 \, dx \right)^{\frac{1}{4}} \leq \\
&\leq c_5 \left(\int\limits_{\{x:x\in\Omega, |u_k(t,s,x)|>|v_k(t,s,x)|\}} |w(t,x)|^6 \, dx \right)^{\frac{1}{4}} \leq \\
&\leq c_6 \left(\int\limits_{\{x:x\in\Omega, |u_k(t,s,x)|>|v_k(t,s,x)|\}} |u_k(t,s,x)|^6 \, dx \right)^{\frac{1}{4}} \leq \\
&\leq c_6 \|\nabla u_k(t,s)\|_{L_2(\Omega)}^{\frac{3}{2}}, \quad s \leq t \leq T_k,
\end{aligned} \tag{5.21}$$

and

$$\begin{aligned}
\|f(w(t)) - f_k(w(t))\|_{L_{\frac{6}{5}}(\Omega)}^{\frac{6}{5}} &\leq c_7 \int\limits_{\{x:x\in\Omega, |w(t,x)|>k\}} |w(t,x)|^6 \, dx \leq \\
&\leq c_8 \left(\int\limits_{\{x:x\in\Omega, |w(t,x)|>k\}} |w(t,x)|^6 \, dx \right)^{\frac{4}{5}} \times \\
&\times \left(\int\limits_{\{x:x\in\Omega, |u_k(t,s,x)|>|v_k(t,s,x)|\}} |w(t,x)|^6 \, dx \right)^{\frac{1}{5}} \leq \\
&\leq c_9 \left(\int\limits_{\{x:x\in\Omega, |w(t,x)|>k\}} |w(t,x)|^6 \, dx \right)^{\frac{4}{5}} \times \\
&\times \left(\int\limits_{\{x:x\in\Omega, |u_k(t,s,x)|>|v_k(t,s,x)|\}} |u_k(t,s,x)|^6 \, dx \right)^{\frac{1}{5}} \leq
\end{aligned}$$

$$\leq c_{10}\left(\int\limits_{\{x:x\in\Omega,|w(t,x)|>k\}}|w(t,x)|^6\,dx\right)^{\frac{4}{5}}\|\nabla u_k(t,s)\|^{\frac{6}{5}}_{L_2(\Omega)},\quad s\leq t\leq T_k,\quad (5.22)$$

for large enough k. On the other hand, since $\mathcal{A}_{\mathcal{H}}$ is compact subset of $\mathcal{H}$ and $(w(t), w_t(t)) \in \mathcal{A}_{\mathcal{H}}$, we have

$$\sup_{t\in R}\int\limits_{\{x:x\in\Omega,|w(t,x)|>k\}}|w(t,x)|^6\,dx\to 0 \text{ as } k\to\infty \quad (5.23)$$

Thus choosing μ small enough, k large enough and taking into account (5.21)-(5.23) in (5.20), we obtain

$$\frac{d}{dt}\widetilde{\Lambda}_{k,s}(t)+\widehat{c}_1\widetilde{\Lambda}_{k,s}(t)\leq\widehat{c}_2\|\nabla v_{kt}(t,s)\|^2_{L_2(\Omega)}\widetilde{\Lambda}_{k,s}(t),\ \ s\leq t\leq T_k,$$

where $\widehat{c}_1$ and $\widehat{c}_2$ are positive constants and $\widetilde{\Lambda}_{k,s}(t)=E(u_k(t,s))+\frac{\mu}{2}\|\nabla u_k(t,s)\|^2_{L_2(\Omega)}$ $+\mu\langle u_{kt}(t,s),u_k(t,s)\rangle$. Now multiplying both sides of the last inequality by $e^{\int_s^t\left[\widehat{c}_1-\widehat{c}_2\|\nabla v_{kt}(\tau,s)\|^2_{L_2(\Omega)}\right]d\tau}$, integrating over $[s,T_k]$ and multiplying both sides of the obtained inequality by $e^{-\int_s^{T_k}\left[\widehat{c}_1-\widehat{c}_2\|\nabla v_{kt}(t,s)\|^2_{L_2(\Omega)}\right]dt}$, we find

$$\widetilde{\Lambda}_{k,s}(T)\leq\widetilde{\Lambda}_{k,s}(s)e^{-\int_s^{T_k}\left[\widehat{c}_1-\widehat{c}_2\|\nabla v_{kt}(t,s)\|^2_{L_2(\Omega)}\right]dt},\ \ s\leq t\leq T_k,$$

which together with (5.7) yields (5.18). □

By Lemma 5.1 and Lemma 5.2, we have $(w(T_{k_0}), w_t(T_{k_0})) \in \mathcal{H}_1$ and

$$\|w_t(T_{k_0})\|_{H^1(\Omega)}+\|w(T_{k_0})\|_{H^2(\Omega)}\leq\widehat{r}_0,$$

where $\widehat{r}_0$ is independent of (w_0, w_1). Now since $w(t, x)$ satisfies (1.1)-(1.3) on $(T_{k_0},\infty)\times\Omega$, with initial data $(w(T_{k_0}),\ w_t(T_{k_0}))$, applying Lemma 4.1 and taking into account the last inequality, we find $(w_0, w_1) \in (H^2(\Omega)\cap H^1_0(\Omega))\times H^1_0(\Omega)$ and

$$\|(w_0,w_1)\|_{H^2(\Omega)\times H^1(\Omega)}\leq R_0,$$

where the positive constant R_0 is independent of (w_0, w_1). So $\mathcal{A}_{\mathcal{H}}$ is a bounded subset of $(H^2(\Omega)\cap H^1_0(\Omega))\times H^1_0(\Omega)$ and that is why it coincides with $\mathcal{A}_{\mathcal{H}_1}$.

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